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Dispersion Analysis by the Simultaneous Application of Tolerances

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Introduction

ONE well-known technique for accomplishing a dispersion analysis is the so-called root-sum-square (RSS) method. Normally, one conducts the RSS method by first calculating a nominal case and then successively varying one parameter at a time, while observing the deviation from nominal caused by the perturbation of that particular parameter. These deviations from nominal are then root-sum-squared to obtain the expected resultant deviation under toleranced conditions. The drawback of this procedure is that it requires n separate analyses for n given parameters.

In this paper it is shown that, under most conditions, it is feasible to perform a single analysis in which the various parameters are toleranced simultaneously, and accomplish essentially the same end result that the RSS method yields.

Conventional Root-Sum-Square (RSS) Approach

Assume that the function Y, which is being investigated for the effect of tolerances, is a function of the parameters $X_1, X_2, \ldots, X_n$. The approximate dispersion in Y due to parameter variations can be expressed by the linear expansion

$$\Delta Y = \sum_{i=1}^{n} K_i \Delta X_i \tag{1}$$

where the K_i terms are usually referred to as "influence coefficients" and the ΔX_i terms represent the parameter deviations from nominal.

Since the X_i parameters are subject to tolerances, they can be considered as random variables that are statistically distributed about their nominal values. If the X_i parameters are independent, then the k-sigma dispersion in ΔY is given by the familiar RSS relationship

$$k\sigma_Y = \left[\sum_{i=1}^{n} (K_i k \sigma_{X_i})^2\right]^{1/2} \tag{2}$$

where the σ_{X_i} values represent the standard deviations of the individual ΔX_i distributions.

To apply Eq. (2), we must first evaluate the K_i influence coefficients. Normally, the relationship between Y and its parameters is known only implicitly; therefore, one usually evaluates the influence coefficients by varying the parameters one at a time and noting the corresponding dispersions in Y. Thus, this procedure requires n separate analyses for n given parameters.

Proposed Approach

Assumptions

In addition to the usual assumptions of linearity and statistical independence required for the RSS method, the proposed approach is predicated on the following assumptions: 1) the directional characteristics of the tolerance effects, i.e., the signs associated with the products of the influence coefficients and their corresponding tolerances, are known; and 2) the significant tolerance effects outnumber the insignificant tolerance effects.

The first assumption is vital to the proposed dispersion analysis technique, since it depends on applying the tolerances such that their effects are all in the same direction. Satisfaction of this assumption may require some previous experience or some a priori reasoning based on the physics of the problem.

The second assumption is aimed at achieving good accuracy, since the accuracy of the method depends on constraining the value of a statistical parameter known as the coefficient of variation† between zero and unity. Should a preponderance of small terms be toleranced in proportion to the significant terms, then it is possible for the coefficient of variation to exceed the latter constraint. From a practical standpoint, however, this assumption will tend to be satisfied automatically, as the analyst will usually exclude negligible effects from the analysis.

Description of method

The approach suggested in this paper is to perform a single analysis and obtain essentially the same result for $k\sigma_Y$ as would normally be obtained from the conventional RSS approach. The procedure is simply to apply appropriately scaled tolerances simultaneously, in the proper direction, and note the resultant dispersion in Y due to the combined effect of the tolerances.

Referring to Eq. (1), note that if all the tolerance effects were in at the same time and in the same direction, say, positive, then all the parameter deviations would contribute uniquely to an increase in ΔY. With this in mind, we can express the resulting dispersion in Y by a modified form of Eq. (1) as

$$\Delta Y = \sum_{i=1}^{n} |K_i \Delta X_i| \tag{3}$$

where the absolute values denote that all the terms are forced positive by proper selections of signs associated with the tolerances.

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† Defined as the ratio of the standard deviation to the mean of a statistical sample.[1]

A resultant dispersion ΔY is sought that represents the same number of standard deviations k that would be obtained from the conventional RSS analysis. Thus, we can let $\Delta Y = k\sigma_Y$. Similarly, the ΔX_i tolerances can be assumed to represent a certain number of standard deviations m (as yet undetermined) of their corresponding distributions; that is, we can let $\Delta X_i = m\sigma_{X_i}$. Equation (3) can now be written as

$$k\sigma_Y = \sum_{i=1}^{n} |K_i m \sigma_{X_i}| \tag{4}$$

Equation (4) represents the basic philosophy of the proposed dispersion analysis method in that the desired result is obtained directly as the result of a single analysis in which all the tolerances are applied simultaneously.

Determination of the tolerance magnitudes

Before the approach indicated by Eq. (4) can actually be utilized, we must first evaluate the factor m, which determines the magnitudes of the tolerances being applied. Equating Eqs. (2) and (4) and solving for m yields

$$m = k\left[\sum_{i=1}^{n} |K_i\sigma_{X_i}|^2\right]^{1/2} \Bigg/ \sum_{i=1}^{n} |K_i\sigma_{X_i}| \tag{5}$$

Examination of Eq. (5) indicates that m is a function of an unknown set of numbers representing the tolerance terms $|K_1\sigma_{X_1}|$, $|K_2\sigma_{X_2}|$, $\ldots$, $|K_n\sigma_{X_n}|$. Considering the sequence $\{|K_i\sigma_{X_i}|\}$ as a random sequence with a mean and a variance given by standard statistical definitions, and defining the sigma/mean ratio as the coefficient of variation V, we can simplify Eq. (5) to

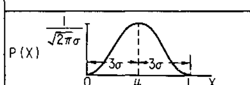
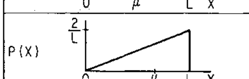
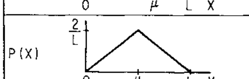
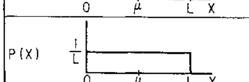
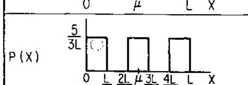
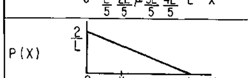
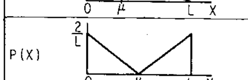
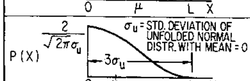
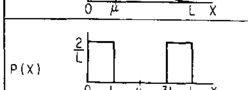
$$m = k[(V^2 + 1)/n]^{1/2} \tag{6}$$

where $V \triangleq \sigma/\mu$. An exact solution for m is actually impossible, since it requires a knowledge of the influence coefficients (which remain unknown under the current approach). However, an approximate solution is found that turns out to be surprisingly accurate under most circumstances.

As shown in the Appendix, the coefficient of variation V is generally constrained between the limits of zero and unity, that is, $0 \leq V \leq 1$. Substituting these limits into Eq. (6) gives the lower and upper bounds on m as $k/n^{1/2} \leq m \leq k(2/n)^{1/2}$. Taking the average of the minimum and maximum values of m as an approximation to the true value of m gives the final result of

$$m \cong 1.2k/n^{1/2} \tag{7}$$

Table 1 Resultant error in proposed tolerance method for various distributions of component tolerance terms

DISTRIBUTION OF X	μ	σ	$V = \frac{\sigma}{\mu}$	$m_{\epsilon,\%}$*
	$\frac{L}{2}$	$\frac{L}{6}$	$\frac{1}{3} = 0.333$	+13.8
	$\frac{2}{3}L$	$\frac{2}{3\sqrt{8}}L$	$\frac{1}{\sqrt{8}} = 0.354$	+13.1
	$\frac{L}{2}$	$\frac{L}{2\sqrt{6}}$	$\frac{1}{\sqrt{6}} = 0.408$	+11.1
	$\frac{L}{2}$	$\frac{L}{2\sqrt{3}}$	$\frac{1}{\sqrt{3}} = 0.577$	+ 3.9
	$\frac{L}{2}$	$\sqrt{\frac{11}{10}}L$	$\sqrt{\frac{11}{5}} = 0.663$	0
	$\frac{L}{3}$	$\frac{L}{3\sqrt{2}}$	$\frac{1}{\sqrt{2}} = 0.707$	− 2.0
	$\frac{L}{2}$	$\frac{L}{2\sqrt{2}}$	$\frac{1}{\sqrt{2}} = 0.707$	− 2.0
	$\frac{L}{3}\sqrt{\frac{2}{\pi}}$	$\frac{L}{3}\sqrt{1-\frac{2}{\pi}}$	$\sqrt{\frac{\pi}{2}-1} = 0.756$	− 4.3
	$\frac{L}{2}$	$\frac{L}{4}\sqrt{\frac{7}{3}}$	$\frac{1}{2}\sqrt{\frac{7}{3}} = 0.764$	− 4.6

*HERE A POSITIVE ERROR DENOTES A CONSERVATIVE RESULT; A NEGATIVE ERROR, AN OPTIMISTIC RESULT.

Table 2 Computation of a 3σ dispersion by the simultaneous tolerance technique

Parameter	Nominal value	Tolerance applied, %	Toleranced value
a_0	1	+4.53	1.0453
a_1	20	+4.53	20.906
a_2	75	−4.53	71.6025
b_0	0.1	+4.53	0.1045
b_1	3	−4.53	2.8641
b_2	27.5	−4.53	26.2542
b_3	75	+4.53	78.3975
$\|G(S)\|_{S=j10}$	0.7071		0.8102
$3\sigma_G = G_{\text{tol}} - G_{\text{nom}} = 0.1031$			

Accuracy

The error in the dispersion computed by the proposed method can be computed from the difference between the approximate value of m given by Eq. (7) and the true value given by Eq. (6). To check the accuracy of the method under typical situations, we computed the error in m for a wide variety of one-sided distributions of the $|K_i\sigma_{X_i}|$ array of terms. (For simplicity, the distributions were assumed continuous rather than discrete.) The results are summarized in Table 1. The error in the proposed method of simultaneous tolerancing turns out to be surprisingly small—on the order of 10%—and appears to be biased in a conservative direction.

Example

Given the transfer function $(a_0S^2 + a_1S + a_2)/(b_0S^3 + b_1S^2 + b_2S + b_3)$ where $S \triangleq$ the Laplace operator, find the 3σ variation in its gain characteristic at $\omega = 10$ rad/sec, assuming the 3σ dispersion in each parameter is 10%.

Applying Eq. (7) with $k = 3$ and $n = 7$ gives an $m = 1.36$, which corresponds to tolerance deviations of 4.53%. Calculation of the nominal and toleranced values of $|G(S)|$ is illustrated in Table 2. (Note that the signs associated with the tolerances were deliberately chosen so that the effects of the tolerances were in the same direction.)

The 3σ dispersion in $|G(s)|$ calculated by the conventional RSS method yielded a result of 0.1049, indicating a difference between the two methods of less than 2% in this particular example.

Appendix: Discussion of the Lower and Upper Bounds of the Coefficient of Variation V of the Tolerance Terms

The fact that the distribution of the $|K_i\sigma_{X_i}|$ terms is one-sided puts a severe constraint on the range of variation of V, as is noted shortly.

The lower bound for the coefficient of variation is immediately apparent if we consider a set of terms having zero standard deviation and finite mean, which, of course, corresponds to a distribution whose terms are all equal. Thus, $V = 0$ establishes the lower bound.

An upper bound can be determined if we consider the distribution that leads to the largest possible value for V. If all the terms are required to satisfy $x_m \leq x_i \leq x_M$ (changing the notation from that used previously for convenience), where x_m is the minimum value of the set and x_M the maximum value, it can be shown by use of the principle of optimality[2] that V is maximized when one of the terms takes on the maximum value and all the remaining terms take on the minimum value. This configuration leads to an upper bound for the coefficient of variation of $(x_M - x_m)(n-1)^{1/2}/[x_M + (n-1)x_m]$.

The aforementioned upper bound, however, is inappropriate for the case at hand, since the distribution that led to it is specifically prohibited on the basis of assumption 2. The worst possible distribution that does not contradict assumption 2 corresponds to one where x_m is zero and where half the terms are equal to x_m and the rest to x_M. This leads to a coefficient of variation of unity, thus establishing an upper bound compatible with the assumptions of the proposed method.

To illustrate the tendency for V to satisfy the constraints previously established, i.e., $0 \leq V \leq 1$, we refer to Table 1 where V varied only from 0.333 to 0.764 for the wide variety of one-sided distributions considered.

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Arc Motion in High-Pressure Cross-Field Devices

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RECENT experiments by the authors with several coaxial cross-field plasma devices have shown some anomalous behavior in the rotational speeds of an arc. Because of the present interest in such devices, pertinent aspects of the experiments are summarized and one explanation is proposed for the observed phenomena.

Three concentric, cross-field configurations designated as "A," "B," and "A"$_{mod}$ are considered. Device "A" is shown schematically in Fig. 1a. It provides for heating and acceleration of the gas within a vortex chamber and is being developed for use with hypervelocity wind tunnels, see Refs. 1 and 2. The "A" devices operate from a direct current power supply with an externally applied magnetic field. Figure 1b is a schematic of the "B" devices. Experiments made with these devices are reported in Ref. 3. Although these latter tests were of a pulsed nature, the current flow as well as the rotational speed of the arc reached steady values before the circuit was interrupted. Two principle differences exist between the "A" and "B" devices. In "A" the cathode surface is a logarithmic spiral of the form $r = ae^{m\Theta}$ and as such provides for a radially outward displacment of the cathode surface. The second difference is the impressed gas flow arrangement. All of the experiments with the "A" devices were made with a mass flow of gas forced radially outward across the vortex. In the "B" devices, no such flow was impressed on the vortex, but gradients established by the vortex were solely responsible for any flow of gas. Figure 1c schematically shows the third device, which is an "A" device modified with a circular cylindrical cathode geometry. In all three devices the motion of the arc was observed by means of light pipes connected to oscilloscopes through emitter follwers circuits.

Figure 2 shows the measured cathode root velocities as a function of the electromagnetic driving parameter BI. The arc current ranged from 1300 amp to 4600 amp. The chamber pressure levels and magnetic field strength are indicated for each set of data. The cathode root velocities for the "A" devices are seen to be anomalously high compared with the velocities measured in the "B" devices. The electrode gap in the "B" devices was changed from 1.27 cm to 2.54 cm with almost no effect on the cathode root velocities. Also, the effect of an impressed radial mass flow was investigated in the "A"$_{mod}$ device by tests made with and without endwalls and with and without a forced mass flow. Little effect was found from these changes, as can be seen in Fig. 2. The conclusion that radial mass flow does not have a significant effect on the vortex would be erroneous, however, because even with no endwalls the vortex establishes a substantial outward radial flow with gas being discharged near the cathode surface. This fact was observed during tests with the endwalls removed. For the tests at the highest current and applied magnetic field strengths, the cathode root velocity in the spiral cathode arrangement is more than four times greater than the root velocity for the circular cathode.

Probable Cause of Anomalous Cathode Root Velocities

Electromagnetic effects offer one explanation for the high cathode root velocities in the spiral geometries. Figure 3 illustrates a simple model of a spiral geometry and shows how the cathode sheath moves radially outward across lines of applied magnetic flux during each rotation of the arc column. Since high ion densities occur near the surface of metal cathodes, the outward radial motion of this sheath constitutes a noticeable "local" current that provides a Lorentz force in

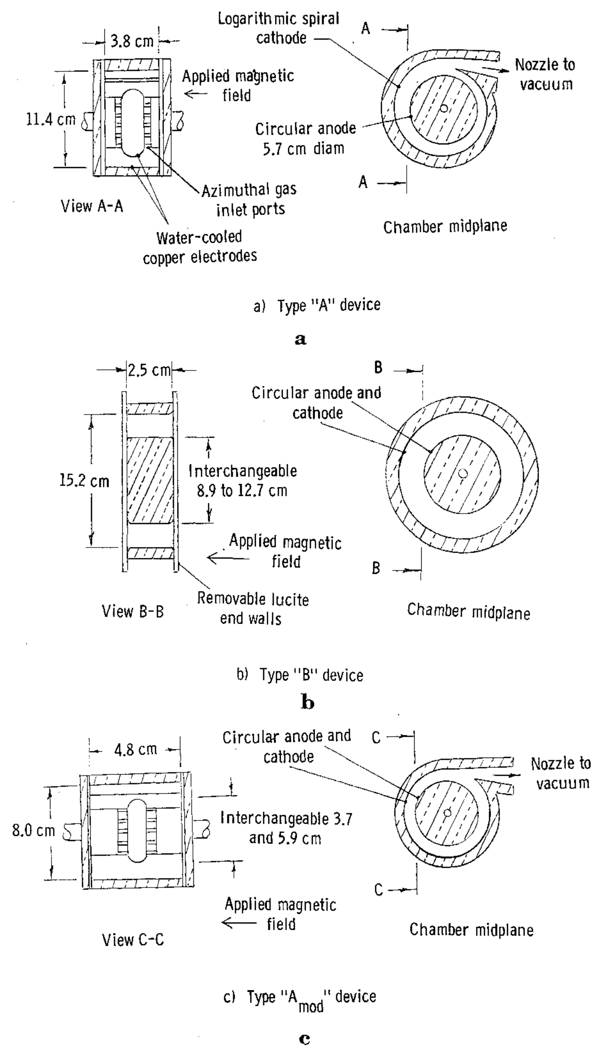

Fig. 1 Experimental crossed field devices.

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